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LETTER TO THE EDITOR

Anharmonic oscillators, spectral determinant and short exact sequence of $U_q(\widehat{\mathfrak{sl}}_2)$

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Abstract. We prove one of the conjectures, raised by Dorey and Tateo (1998 Anharmonic oscillators, the thermodynamic Bethe ansatz, and nonlinear integral equations *Preprint* DTP-98/81, ITPA 98-41, (hep-th/9812211)) in the connection among the spectral determinant of anharmonic oscillator and vacuum eigenvalues of transfer matrices in field theory and statistical mechanics. The exact sequence of $U_q(\widehat{\mathfrak{sl}}_2)$ plays a fundamental role in the proof.

Recently, Dorey and Tateo have found a remarkable connection among the spectral determinants of a 1D Schrödinger operator associated with the anharmonic oscillator, transfer matrices and $\boldsymbol{Q}$ operators in CFT for a certain value of Virasoro parameter p [1]. This has been subsequently generalized to general values of p by appropriate modifications on the Hamiltonian [2]. The most fundamental equalities among parity-dependent spectral determinants and $\boldsymbol{Q}_\pm$ operators are proven by utilizing the quantum Wronskian relation.

In this letter, we provide an elementary proof of the conjectures in [1] concerning the sum rule which is closed only among the spectral determinant (= product of parity dependent spectral determinants). The short exact sequence in quantum affine Lie algebra $U_q(\widehat{\mathfrak{sl}}_2)$ plays a fundamental role. We consider the Schrödinger equation,

$$\hat{H}\Psi_k(x) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^{2M}\right)\Psi_k(x) = E_k\Psi_k(x). \tag{1}$$

Here M is assumed to be an integer greater than two.

The spectral problem associated with this has been scrutinized in [8–12]. The properties can be encoded into the spectral determinant

$$D_M(E) = \det(E + \hat{H}) = D_M(0)\prod_{k=0}^{\infty}\left(1 + \frac{E}{E_k}\right) \tag{2}$$

and $D_M(0) = 1/\sin(\pi/(2M+2))$.

In the following, we adopt a notation $\mathcal{D}_M(x) := D_M(\mathrm{e}^{\pi x/(M+1)})$.

Remarkably, it satisfied the exact functional relation [9], which reduces to a simple polynomial form for $M = 2$:

$$\mathcal{D}_2(x)\mathcal{D}_2(x+2\mathrm{i})\mathcal{D}_2(x+4\mathrm{i}) = \mathcal{D}_2(x) + \mathcal{D}_2(x+2\mathrm{i}) + \mathcal{D}_2(x+4\mathrm{i}). \tag{3}$$

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For $M > 2$, such a simple polynomial expression is not available and it reads explicitly,

$$\sum_{k=0}^{M} \phi(x+2\mathrm{i}k) = \frac{\pi}{2}$$
$$\phi(x) = \arcsin \frac{1}{\sqrt{\mathcal{D}_M(x)\mathcal{D}_M(x+2\mathrm{i})}}. \tag{4}$$

On the other hand, transfer matrices are introduced in the analysis of statistical mechanics [3], integrable structures in $c < 1$ CFT [4, 5] and so on†. We do not specify its precise definition. (We refer interested readers to [3, 7, 4]) For our purpose, the following facts are sufficient. Let the deformation parameter q be $\mathrm{e}^{\mathrm{i}\pi\beta^2}$. We denote a $U_q(\widehat{\mathfrak{sl}_2})$ module $W_j(\lambda)$, which corresponds to the $(j+1)$-dimensional module of $U_q(\mathfrak{sl}_2)$. The associated ('Drinfel'd') polynomial is given by

$$P(\lambda') = (1-q^{j-1}\lambda\lambda')(1-q^{j-3}\lambda\lambda')\ldots(1-q^{-j+1}\lambda\lambda'). \tag{5}$$

See [6] for precise definitions. Taking trace of monodromy operator over $W_j(\lambda)$, one can define the transfer matrix $\boldsymbol{T}_j(\lambda)$. $\boldsymbol{T}_j(\lambda)$ constitutes a commutative family and satisfy the 'T-system',

$$\boldsymbol{T}_j(q\lambda)\boldsymbol{T}_j(q^{-1}\lambda) = \boldsymbol{I} + \boldsymbol{T}_{j+1}(\lambda)\boldsymbol{T}_{j-1}(\lambda) \qquad j = 1, 2, \ldots \tag{6}$$

and $\boldsymbol{T}_0 = \boldsymbol{I}$. (Note the suffix j and the normalization of λ are defined differently from [4].)

As we are considering these operators on their common eigenvector space, we will use the same symbol $\boldsymbol{T}_j$ for its eigenvalue.

For $\beta^2 = \frac{1}{M+1}$, the above functional relations close finitely due to the following property:

$$\boldsymbol{T}_{M-j}(\lambda) = \boldsymbol{T}_{M+j}(\lambda) \qquad j = 1, \ldots, M \tag{7}$$

and $\boldsymbol{T}_{2M+1}(\lambda) = 0$.

Again we adopt the 'additive variable' x rather than 'multiplicative variable' λ, $T_j(x) = \boldsymbol{T}_j(\mathrm{e}^{\pi x/(M+1)})$. Then the T-system (6) reads

$$T_j(x+\mathrm{i})T_j(x-\mathrm{i}) = 1 + T_{j+1}(x)T_{j-1}(x). \tag{8}$$

We also remark periodicity,

$$T_j(x+(2M+2)\mathrm{i}) = T_j(x). \tag{9}$$

(The variable θ in [1] is related to x by $\theta = x\pi/2M$.)

In [14, 7], it has been shown that the substitution of $Y_j(x) = T_{j-1}(x)T_{j+1}(x)$ into (8) yields the well known Y-system [15]. The solution to the Y- or T-system is not necessarily unique. One needs to know zeros or singularities of $Y_j(x)$, or equivalently, $T_j(x)$ in a 'physical strip' ($\mathrm{Im}\, x \in [-1, 1]$) to fix a solution. With this knowledge, one reaches the thermodynamic Bethe ansatz (TBA) equation, which yields a unique solution.

Dorey and Tateo showed, for $M = 2$, $\mathcal{D}_2(x)$ and $T_2(x)$ satisfy the same functional relation (3). The coincidence carries forward. With some additional tuning of parameters, they share the same analytic structure, which validates $\mathcal{D}_2(x) = T_2(x)$. For $M > 2$ they presented numerical evidences to support a conjecture $\mathcal{D}_M(x) = T_M(x)$ instead of proving that they satisfy the same functional relation (4).

In the following we will supply the proof. The idea is to utilize the short exact sequence of $U_q(\widehat{\mathfrak{sl}_2})$ in [6]. (The T-system is one of the simplest consequences of it.) The short exact sequence reads,

$$0 \longrightarrow W_{\alpha-p}(\lambda q^{-p}) \otimes W_{\beta-p}(\lambda' q^{-p}) \longrightarrow W_\alpha(\lambda) \otimes W_\beta(\lambda')$$
$$\longrightarrow W_{p-1}(\lambda q^{\alpha-p+1}) \otimes W_{\alpha+\beta-p+1}(\lambda' q^{-(\alpha-p+1)}) \longrightarrow 0$$
$$\text{for} \quad \frac{\lambda'}{\lambda} = q^{\alpha+\beta-2p+2}. \tag{10}$$

† See the discussions on transfer matrices in quantum impurity problems [13].

We abbreviate these modules to $W_0 \sim W_5$, and the corresponding transfer matrix T_{W_i} (trace of monodromy operator over W_i). Then the consequence of (10) is,

$$0 = T_{W_0}T_{W_1} - T_{W_2}T_{W_3} + T_{W_4}T_{W_5}. \tag{11}$$

In the additive variable, the equivalent 'generalized T-system' reads,

$$\begin{aligned} T_\alpha(x)T_\beta(x + (\alpha + \beta - 2p + 2)\mathrm{i}) &= T_{\alpha-p}(x - \mathrm{i}p)T_{\beta-p}(x + \mathrm{i}(\alpha + \beta - p + 2)) \\ &\quad + T_{p-1}(x + \mathrm{i}(\alpha - p + 1))T_{\alpha+\beta-p+1}(x + \mathrm{i}(\beta - p + 1)). \end{aligned} \tag{12}$$

One substitutes $\alpha = \beta = p = j$ to recover (8). We refer to the above identity by $I(\alpha, \beta, p, x)$.

We first give the following statement.

Theorem 1. *Let $\psi(x)$ be $\phi(x)$ in (4) replacing $\mathcal{D}_M(x)$ by $T_M(x)$,*

$$\psi(x) = \arcsin \frac{1}{\sqrt{T_M(x)T_M(x + 2\mathrm{i})}}. \tag{13}$$

Then we have,

$$\sum_{k=0}^{M} \psi(x + 2\mathrm{i}k) = \frac{\pi}{2}. \tag{14}$$

We prove the above theorem in an equivalent form,

$$\begin{aligned} &\cos(\psi(x) + \psi(x + 2\mathrm{i}) + \cdots + \psi(x + (2M - 4)\mathrm{i})) \\ &\qquad = \sin(\psi(x + (2M - 2)\mathrm{i}) + \psi(x + 2M\mathrm{i})) \qquad M \text{ odd} \\ &\cos(\psi(x) + \psi(x + 2\mathrm{i}) + \cdots + \psi(x + (2M - 2)\mathrm{i})) = \sin(\psi(x + 2M\mathrm{i})) \qquad M \text{ even} \end{aligned} \tag{15}$$

following [9]. To be precise, the condition (15) literally leaves multiples of 2π indeterminate in the right-hand side of (14). This can however be fixed from the asymptotic value $T_M(|x| \to \infty) = 1/\sin(\pi/(2M + 2))$, which can be derived from the algebraic relation (8) by sending $x \to \infty$. We verify that (15) coincides with (14).

To show (15), we prepare a few lemmas as follows.

Lemma 1.

$$\begin{aligned} \sin(\psi(x) + \psi(x + 2\mathrm{i})) &= \frac{T_1(x + \mathrm{i}(M + 3))}{\sqrt{T_M(x)T_M(x + 4\mathrm{i})}} \\ \cos(\psi(x) + \psi(x + 2\mathrm{i})) &= \frac{T_{M-2}(x + 2\mathrm{i})}{\sqrt{T_M(x)T_M(x + 4\mathrm{i})}}. \end{aligned} \tag{16}$$

Proof. We first note

$$\begin{aligned} \cos(\psi(x)) = \sqrt{1 - \sin^2(\psi(x))} &= \sqrt{1 - \frac{1}{T_M(x)T_M(x + 2\mathrm{i})}} \\ &= \sqrt{\frac{T_{M-1}(x + \mathrm{i})T_{M+1}(x + \mathrm{i})}{T_M(x)T_M(x + 2\mathrm{i})}} = \frac{T_{M-1}(x + \mathrm{i})}{\sqrt{T_M(x)T_M(x + 2\mathrm{i})}} \end{aligned} \tag{17}$$

where (8) and (7) are used in the last two equalities. By expanding the left-hand side of the first equation in lemma 1, we have,

$$\begin{aligned} \sin(\psi(x) + \psi(x + 2\mathrm{i})) &= \sin(\psi(x))\cos(\psi(x + 2\mathrm{i})) + \sin(\psi(x + 2\mathrm{i}))\cos(\psi(x)) \\ &= \frac{T_{M-1}(x + 3\mathrm{i}) + T_{M-1}(x + \mathrm{i})}{T_M(x + 2\mathrm{i})\sqrt{T_M(x)T_M(x + 4\mathrm{i})}} \\ &= \frac{T_1(x + \mathrm{i}(M + 3))}{\sqrt{T_M(x)T_M(x + 4\mathrm{i})}} \end{aligned} \tag{18}$$

where we have applied $I(M, 1, 1, x+2\mathrm{i})$,

$$\begin{aligned} T_M(x+2\mathrm{i})T_1(x+\mathrm{i}(M+3)) &= T_{M-1}(x+\mathrm{i}) + T_{M+1}(x+3\mathrm{i}) \\ &= T_{M-1}(x+\mathrm{i}) + T_{M-1}(x+3\mathrm{i}). \end{aligned} \tag{19}$$

The second relation is similarly proved. □

We further generalize the expansion of trigonometric functions with even arguments more than two.

Lemma 2. *Let ℓ be an odd integer. We have the following relation:*

$$\begin{aligned} \begin{pmatrix} \cos(\psi(x)+\psi(x+2\mathrm{i})+\cdots+\psi(x+2\ell\mathrm{i})) \\ \sin(\psi(x)+\psi(x+2\mathrm{i})+\cdots+\psi(x+2\ell\mathrm{i})) \end{pmatrix} &= \frac{1}{\sqrt{T_M(x)T_M(x+\mathrm{i}(2\ell+2))}} \\ &\quad\times \prod_{k=1}^{(\ell-1)/2} \frac{\mathfrak{L}(x+(4k-4)\mathrm{i})}{T_M(x+4k\mathrm{i})} \begin{pmatrix} T_{M-2}(x+2\ell\mathrm{i}) \\ T_1(x+\mathrm{i}(M+2\ell+1)) \end{pmatrix} \end{aligned} \tag{20}$$

$$\mathfrak{L}(x) := \begin{pmatrix} T_{M-2}(x+2\mathrm{i}), & -T_1(x+\mathrm{i}(M+3)) \\ T_1(x+\mathrm{i}(M+3)), & T_{M-2}(x+2\mathrm{i}) \end{pmatrix}$$

where the order of the operator product should be understood as,

$$\mathfrak{L}(x)\mathfrak{L}(x+4\mathrm{i})\ldots\mathfrak{L}(x+(2\ell-6)\mathrm{i}).$$

Proof. This is easily shown by iterative applications of the recursion relation,

$$\begin{aligned} \begin{pmatrix} \cos(\psi(x)+\psi(x+2\mathrm{i})+\cdots+\psi(x+2\ell\mathrm{i})) \\ \sin(\psi(x)+\psi(x+2\mathrm{i})+\cdots+\psi(x+2\ell\mathrm{i})) \end{pmatrix} &= \frac{1}{\sqrt{T_M(x)T_M(x+4\mathrm{i})}} \\ &\quad\times\mathfrak{L}(x) \begin{pmatrix} \cos(\psi(x+4\mathrm{i})+\psi(x+6\mathrm{i})+\cdots+\psi(x+2\ell\mathrm{i})) \\ \sin(\psi(x+4\mathrm{i})+\psi(x+6\mathrm{i})+\cdots+\psi(x+2\ell\mathrm{i})) \end{pmatrix} \end{aligned} \tag{21}$$

which follows from lemma 1. □

The above recursion procedure is regarded as the forward propagation. Next let us perform the back-propagation procedure: we apply matrices $\mathfrak{L}$ on the column vector. We observe a simple pattern there, which can be summarized as the following lemma.

Lemma 3. *We introduce a vector v_t by*

$$v_t := \begin{pmatrix} T_{M-2-2t}(x-(6+2t)\mathrm{i}) \\ T_{2t+1}(x+(M-5-2t)\mathrm{i}) \end{pmatrix}. \tag{22}$$

Then the following back-propagation relation holds,

$$\mathfrak{L}(x+\mathrm{i}(2M-10-4t))v_t = T_M(x-(4t+8)\mathrm{i})v_{t+1}. \tag{23}$$

Proof. The first component in the left-hand side in (23) reads

$$\begin{aligned} &T_{M-2}(x-(10+4t)\mathrm{i})T_{M-2-2t}(x-(6+2t)\mathrm{i}) \\ &\qquad -T_1(x+\mathrm{i}(M-9-4t))T_{2t+1}(x-\mathrm{i}(M+2t+7)) \end{aligned} \tag{24}$$

where we have applied the periodicity (9) to the last component. By the use of $I(M-2, M-2t-2, M-2t-3, x-(4t+10)\mathrm{i})$, one finds (24) equals $T_M(x-(4t+8)\mathrm{i})T_{M-2-2(t+1)}(x-\mathrm{i}(2(t+1)+6)\mathrm{i})$, which is nothing but the first component of the right-hand side. Similarly one applies $I(M, 2t+3, 2, x-(4t+8)\mathrm{i})$ to the second component of the right-hand side in (23), leading to the equality. □

We shall fix the relation between ℓ and M as follows:

$$\ell = \begin{cases} M-2 & \text{if } M = \text{odd} \\ M-1 & \text{if } M = \text{even}. \end{cases} \tag{25}$$

Then our final lemma is as follows.

Lemma 4. *Under the above relation between ℓ and M, one has*

$$\cos(\psi(x) + \cdots \psi(x + 2\ell \mathrm{i})) = \begin{cases} \dfrac{T_1(x - \mathrm{i}(M+3))}{\sqrt{T_M(x) T_M(x + (2M-2)\mathrm{i})}} & M \text{ odd} \\ \dfrac{1}{\sqrt{T_M(x) T_M(x + 2M\mathrm{i})}} & M \text{ even}. \end{cases} \tag{26}$$

Proof. Let us apply $\mathfrak{L}$ to the vector in (20). For M odd, the initial vector reads,

$$\begin{pmatrix} T_{M-2}(x - 6i) \\ T_1(x + \mathrm{i}(M-5)\mathrm{i}) \end{pmatrix} \tag{27}$$

which is nothing but $v_{t=0}$ in (22). The product of matrices in (20) is

$$\mathfrak{L}(x)\mathfrak{L}(x+4\mathrm{i}) \ldots \mathfrak{L}(x + (2M-14)\mathrm{i})\mathfrak{L}(x + (2M-10)\mathrm{i}). \tag{28}$$

Thus one can apply (23) recursively to find

$$\mathfrak{L}(x) \ldots \mathfrak{L}(x + (2M-10)\mathrm{i})v_0 = \prod_{j=1}^{(M-3)/2} T_M(x - (4+4j)\mathrm{i}) \begin{pmatrix} T_1(x - \mathrm{i}(M+3)) \\ T_{M-2}(x - 2\mathrm{i}) \end{pmatrix}. \tag{29}$$

Substituting (29) into (20), and after rearrangement using (9) one arrives at the odd case of lemma 4 from the first component. For M even, initial vector is $v'_{t=0} = v_{t=0}(x \to x + 2\mathrm{i})$. Similarly, the product of $\mathfrak{L}$ is given by $x \to x + 2\mathrm{i}$ in (28). The result of the application reads

$$\cdots \mathfrak{L}(x + (2M-12)\mathrm{i})\mathfrak{L}(x + (2M-8)\mathrm{i})v'_0$$
$$= \prod_{j=1}^{(M-2)/2} T_M(x - (2+4j)\mathrm{i}) \begin{pmatrix} T_0(x - (M+2)\mathrm{i}) \\ T_{M-1}(x - \mathrm{i}) \end{pmatrix}. \tag{30}$$

Again the substitution of (30) into (20) leads to lemma 4 for M even case. □

Proof of theorem 1. Now the left-hand side of (15) is explicitly written in terms of T-functions in lemma 4. It remains to check that it coincides with right-hand side. This can be easily done by (1) or from the definition of $\psi(x)$ itself. □

As is noted previously, the common functional relation does not grantee the equality, $T_M(x) = \mathcal{D}_M(x)$: one needs further knowledge on their analytic structures. In this respect, we shall entirely depend on the argument in [1]. In the TBA equation originated from T-system, one shall take the massless drive terms, $m_a r \mathrm{e}^{\pi x/2M}$, $(a = 1, \ldots 2M-1)$ and setting $m_M r = \pi^{1/2}\Gamma(\frac{1}{2M})/(M\Gamma(\frac{3}{2} + \frac{1}{2M}))$. Then $\mathcal{D}_M(x)$ and $T_M(x)$ share the same analytical properties: both of them have the same 'asymptotic value' and have zeros on $\mathrm{Im}\, x = \pm(M+1)$. The latter is consistent with a property of the Schrödinger operator that eigenstates are all bounded so $E_k > 0$. Thus one concludes the equality, $\mathcal{D}_M(x) = T_M(x)$.

Summarizing, we have proven one of conjectures in [1] that $T_M(x)$ actually shares the same functional relation with $\mathcal{D}_M(x)$. The proof utilizes the exact sequence of $U_q(\widehat{\mathfrak{sl}_2})$. This makes us expect further deep connections between the anharmonic oscillator and quantum integrable structures.

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References

[1] Dorey P and Tateo R 1998 Anharmonic oscillators, the thermodynamic Bethe ansatz, and nonlinear integral equations *Preprint* DTP-98/81, ITPA 98-41, (hep-th/9812211)
[2] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1998 Spectral determinants for Schrödinger equation and Q-operators of conformal field theory *Preprint* hep-th/9812247
[3] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[4] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1997 *Commun. Math. Phys.* **177** 381–98
[5] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1997 *Commun. Math. Phys.* **190** 247–78
[6] Chari V and Pressly A 1991 *Commun. Math. Phys.* **142** 261–83
[7] Kuniba A, Nakanishi T and Suzuki J 1994 *Int. J. Mod. Phys.* A **9** 5215–66
[8] Voros A 1983 *Ann. Inst. Henri Poincaré* A **39** 211–338
[9] Voros A 1992 *Adv. Stud. Pure. Math.* **21** 327–58
[10] Voros A 1994 *J. Phys. A: Math. Gen.* **27** 4653–61
[11] Voros A 1997 *Quasiclassical Method (IMA Proceedings, Minneapolis 1995) (IMA Series vol 95)* ed J Rauch and B Simon (Berlin: Springer) pp 189–224
[12] Voros A 1999 *J. Phys. A: Math. Gen.* **32** 1301–11
[13] Fendley P, Lesage F and Saleur H 1996 *J. Stat. Phys.* **85** 211
[14] Klümper A and Pearce P A 1992 *Physica* A **183** 304
[15] Zamolodchikov Al B 1991 *Phys. Lett.* B **253** 391–4